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RESEARCH MEMORANDUM

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RESEARCH MEMORANDUM

NOTES ON LINEAR PROGRAMMING: PART XII
A COMPOSITE SIMPLEX ALGORITHM--II

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SUMMARY

The standard algorithm of the revised simplex method—with which the reader is assumed to be familiar—has been designed to provide the simplest possible set of rules for solving a linear programming problem in the general case where nothing is known or assumed about the rank of the system. Experience indicates that this simplicity of operation is sometimes gained at the expense of several iterations which tend to replace certain vectors over and over. It is believed that this tendency can be reduced by a more critical analysis of the objectives of the procedure on each iteration. While this complicates the rules of operation somewhat, the latter is not a serious obstacle for large, high-speed computers.

Previous papers involving the use of a dual algorithm have suggested that much information is inherent in a problem which is not normally taken advantage of. The present proposal uses a combination of the normal and dual algorithms, with some modifications, and keeps separate the three essentially distinct ideas with which the simplex method is concerned, namely: non-singularity of a basis matrix, feasibility of the solution, and optimality of the solution. It is believed that an optimal solution will be obtained in fewer iterations by this method.

A COMPOSITE SIMPLEX ALGORITHM—II

Wm. Orchard-Hays

I—NOTATION AND BACKGROUND.

A linear programming problem is concerned with minimizing (or maximizing, which merely involves change of signs) a linear form

(1) $\sum_{j=1}^{n} a_{oj}\, x_j = \min$

subject to the conditions that

(2) $\sum_{j=1}^{n} a_{ij}\, x_j = b_i \quad , \quad i = 1,2,\ldots,m$

(3) $x_j \geq 0 \quad , \quad j = 1,2,\ldots,n,$

where we assume (without loss of generality) that all $b_i \geq 0$.

In the revised simplex method (and in particular here, the form of the method using the product form for the inverse of the basis), the statements (1), (2) are combined in a single matrix equation formed of the following columns (denoted by braces):

$$P_o = \{1, 0, \ldots, 0\}$$
$$P_j = \{a_{oj}, a_{1j}, \ldots, a_{mj}\} \quad , \qquad j = 1,2,\ldots,n$$
$$Q = \{0, b_1, \ldots, b_m\}$$
$$X = \{x_o, x_1, \ldots, x_n\}$$

which are combined to form the matrix $P = [P_o, P_1, \ldots, P_n]$ and the equation

(4) $PX = Q$

whence x_o is to be maximized subject to (3) and (4).

The simplex method works solely with basic solutions. If P does not contain the (m+1)-order identity matrix I, then a basic feasible solution is not in general available to start the iterative procedure for maximizing x_o. That is, what is desired is the explicit knowledge of the following quantities:

(5a) a basis for (m+1)-space chosen from the P_j and denoted by

$$B = [P_{j_o}, P_{j_1}, \ldots, P_{j_m}] \quad , \quad P_{j_o} = P_o \ ;$$

(5b) its associated solution vector $V = \{v_o, v_1, \ldots, v_m\}$ such that $BV = Q$, $v_i \geq 0$ for $i > 0$; and

(5c) the inverse of B whose elements we denote by β_{ik}, the rows by $\beta_i = (\beta_{io}, \beta_{i1}, \ldots, \beta_{im})$, $i = 0,1,\ldots,m$, and the columns, when necessary, by $C_k = \{\beta_{ok}, \beta_{1k}, \ldots, \beta_{mk}\}$, $k = 0,1,\ldots,m$; that is, briefly,

$$B^{-1} = (\beta_{ik}) = \{\beta_o, \beta_1, \ldots, \beta_m\}$$

where the last braces signify a column of rows.

Besides the lack of knowledge of the quantities (5), it is in general not even known whether or not the matrix P has rank m+1, that is, whether or not it is possible to choose a basis for (m+1)-space from the columns of P. At the same time, if certain of the equations (2) were redundant (i.e., rank of P less than m+1), this would not preclude the possibility of a solution to the problem,

even perhaps a unique one. Furthermore, even if the quantities (5) were known, it would be awkward to start the iterative procedure of the simplex method using the product form of B^{-1} in a way to take advantage of this knowledge. To overcome all these difficulties at one fell swoop, "phase one" of the method was devised to provide a convenient, practicable starting point from which could be constructed the necessary transformations which would lead to a basic, feasible solution or show that none existed. It is assumed that the reader is familiar with this procedure and with the use of the product form for B^{-1}

$$B^{-1} = E_k\, E_{k-1} \cdots E_1$$

where the E_ℓ are elementary column matrices. [1,3,5,6]

II—DISADVANTAGES OF PHASE ONE OF THE SIMPLEX METHOD.

Although phase one provides a theoretically fool-proof[1] way of finding a solution to (3) and (4)—or of showing that no solution exists which, in a sense, is a solution to the whole problem—still it introduces difficulties of its own. In the first place, experience has shown that phase one is apt to be unduly long when P has certain types of structures, especially when Q contains many zeros, a condition which arises fairly often. This is costly in computing time and also introduces considerable round-off error into the

[1] It is literally "fool-proof," at least by present knowledge, only if some perturbation of Q is used to guarantee convergence, as for example in the Generalized Simplex Method. [1,3] Throughout this paper this difficulty will be ignored, as it has been in practice for some time. [5,6]

computations before the actual maximization of x_o is begun. Secondly, there is no guarantee that the feasible basis obtained at the end of phase one may not be about as far from optimal as possible. The algorithm to be herein presented is the second of two proposals [8] for overcoming the second of these objections. The first objection is closely related to the difficulty mentioned in footnote 1—that is, although non-convergence appears to be extremely rare, slow convergence is fairly common for phase one. However, it is believed that even this difficulty will be improved by the present algorithm.

III—A NEW LOOK AT ARTIFICIAL VARIABLES AND THE PRODUCT FORM OF B^{-1}.

In the usual phase one, an "artificial" identity matrix is adjoined to P and artificial variables are associated with these unit vectors, an auxiliary maximizing form then being introduced for the purpose of eliminating the artificial vectors or at least of driving the sum of the artificial variables to zero. (This auxiliary form consists of a redundant equation incorporating an additional variable similar to x_o.) Also when the product form of B^{-1} is used, this is usually considered as merely a computationally convenient way of recording B^{-1}. We now wish to look at these variables and the elementary matrices whose product is B^{-1} from slightly—but importantly—different viewpoints.

It is mandatory that we maintain—at all stages of the simplex process—a basis for (m+1)-space, regardless of whether it is wholly or partly artificial, or completely contained in P. Thus when we speak of a "basic, feasible solution" to (3) and (4):

(a) "solution" means that equation (4) is satisfied, and

(b) "feasible" means that condition (3) is satisfied, whereas

(c) "basic" means that for all $x_j > 0$, the associated P_j are linearly independent (and independent of P_o) and hence if there are $p < m$ such P_j $(j > 0)$ then $m-p$ unit vectors must "fill out" the basis.

Clearly these three notions are distinct and the idea of a solution being "basic" is really extraneous to the linear programming problem. Furthermore,

(d) "optimality" refers to (1)

and is still a fourth distinct notion. Although the simplex criterion for optimality requires, in practice, the inverse of a basic, the degree of artificiality is not a consideration. The essence of the present proposal is to keep these four notions distinct but to work on all of them simultaneously.

Since we must always provide a basis, it is natural (and very convenient) to start with the simplest of all bases, the identity matrix I. We will not think of this as an "artificial basis" for the problem but simply as part of the necessary mechanism on which the operation of the method depends. We can obtain feasibility initially by setting all $x_j = 0$, $j = 0,1,\ldots,n$. However, this does not provide a solution to (4) so it is necessary to modify (4) so that the right-hand side is null. For this purpose we introduce m auxiliary non-negative variables $u_i \geq 0$, $i = 1,2,\ldots,m$.[2] We then

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[2]If P contains some unit vectors U_i other than P_o, then the x_i for these vectors can be set equal to b_i initially and no u_i need be introduced for this set.

modify (2) to

$$(6) \quad \sum_{j=1}^{n} a_{ij} x_j = b_i - u_i \quad , \quad i = 1,2,\ldots,m,$$

whence setting $u_i = b_i$ provides a solution to the modified equation (4) $PX = Q - U$ where $U = \{0, u_1, u_2, \ldots, u_m\}$.

We then seek both to maximize x_o and to reduce U to nullity while maintaining feasibility and continuing to provide a basis with which to operate.[3]

Clearly if P contains any non-null vector besides P_o, some P_j $(j > 0)$ can be introduced into the basis eliminating one of the unit vectors U_i, $i > 0$. The choice of which P_j to introduce will be made to depend on the optimizing form (1) as long as it is possible to improve x_o, whereas the choice of which U_i to drop will be such as to maintain feasibility (i.e., keep (3) satisfied) and also maintain all $u_i \geq 0$. The first substitution must reduce some u_i to zero (unless certain u_i were not introduced; see footnote 2) when the corresponding U_i is eliminated from B. Whenever this happens, the u_i is then not allowed to re-enter the problem; that is, when U_i is eliminated from B, u_i is removed from U and neither are ever considered again. However, some u_i may formally remain in U at zero level. Whenever, in the sequel, we refer to $\sum u_k$, the sum is considered as taken over those u_k still formally in U.

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[3]The variables just introduced were suggested by a similar device proposed by E. M. L. Beale and elaborated by Dantzig [8] in which a variable measuring the sum of the deviations from Q was introduced in the right member of the redundant equation usually employed in phase one. Note that we have not used such an equation.

One of the U_i is not necessarily eliminated on each iteration; as always, certain P_j may be introduced into B and replaced later by other P_j. However, the <u>choice</u> of a P_j to introduce depends on (1) and not on an auxiliary form.

Each iteration ℓ will produce an elementary matrix E_ℓ which transforms the inverse of the $(\ell-1)^{st}$ basis $B_{\ell-1}$ to the inverse of the ℓ^{th} basis:

$$B_\ell^{-1} = E_\ell B_{\ell-1}^{-1} = E_\ell E_{\ell-1} \cdots E_1 E_o \quad , \quad E_o = I = B_o^{-1} .$$

If and when a feasible solution to (3) and (4) is obtained, say at the N^{th} iteration, then the associated basis has an inverse

$$(7) \qquad B_N^{-1} = E_N E_{N-1} \cdots E_\ell \cdots E_1$$

where $B_N V_N = Q$ and for any U_i (not in P) remaining in B_N, $v_i = 0$. Hence we can think of the N iterations as essentially computing the E_ℓ whose product gives the inverse of a feasible basis B_N. Note that a feasible basis may be partly "artificial." On the other hand, a basis chosen completely from columns of P may not provide a feasible solution to (3),(4). Hence the notion of a "feasible basis" is really a composite idea.

IV—REQUIRED MODIFICATION OF THE FIRST SIMPLEX CRITERION

<u>Choice A: (Some $\delta_j < 0$)</u>

There are two main criteria which together with Gaussian elimination essentially constitute the simplex method. First, given a basis B, the first row of B^{-1}, β_o, is used as a "pricing vector."

The inner products

(8) $\delta_j = \beta_o P_j$, $j = 1,2,\ldots,n$

are formed. If any $\delta_j < 0$, then an index s is chosen by

(9) $\delta_s = \min_j \delta_j < 0$,

the usual convention being that the smallest such index is chosen in case of ties. If (9) applies, then the vector P_s is introduced in the amount $\theta \geq 0$. The new value x_o^* of the new solution, obtained when P_s replaces (if possible) some vector in B, is given by

(10) $x_o^* = x_o - \theta\delta_s \geq x_o$.

This same rule will be used in the present method whenever (9) applies, i.e., whenever some $\delta_j < 0$. We will call this selection criterion "Choice A."

Choice B: (All $\delta_j \geq 0$, all $u_i = 0$)

In the usual simplex procedure, the condition that all $\delta_j \geq 0$ implies that an optimal solution has been obtained, namely the basis B and its associated solution vector $V = B^{-1}Q$. This implication rests on the following two assumptions:

(i) V has been maintained feasible on every iteration;

(ii) either B contains no artificial vectors or, if it does, the sum of the corresponding artificial variables is zero.

While we will maintain feasibility in the extended sense that (2) holds and also all $u_i \geq 0$, we will certainly have artificial unit vectors in B for at least the first m-u iterations (where u is the

number of unit vectors among the P_j, $j > 0$) and the corresponding u_i need not be zero. Hence we can not claim optimality because all $\delta_j \geq 0$ unless at the same time all $u_i = 0$. The last eventuality is equivalent to the condition of optimality in the normal simplex procedure, and hence we terminate the procedure in such a case since an optimum, feasible solution has been attained.

Choice C: (All $\delta_j \geq 0$, some $u_i > 0$)

When all $\delta_j \geq 0$, we can obtain no improvement in x_o without violating feasibility (cf. (10)), but if also some $u_i > 0$, then we must yet attempt to reduce these u_i to zero in order to claim a solution to (4). For this purpose, we borrow the criterion (slightly modified) from the dual simplex algorithm. [2,7]

Since in this case we wish to reduce the $u_i > 0$, it seems logical to work first on the largest one. Hence we choose the index t by

(11) $u_t = \max u_i > 0$,

(taking the smallest such index, say, in case of ties) and then use β_t to form the inner products $\beta_t P_j$. If we succeed in eliminating U_t with some P_j—that is, if we can determine a P_s and replacing U_t with P_s does not violate feasibility—then $\beta_t P_s = y_t = y_r$ will be the "pivot element" for the elimination. (Since $Y_s = B^{-1}P_s$, $y_i = \beta_i P_s$.) Then also, we will have $\theta_r = u_t/y_t = x_s$ and this must be non-negative, so we must have $y_t = \beta_t P_s > 0$. Furthermore, although we cannot increase x_o, we want to reduce it as little as possible. Since the

change in x_o is $\theta_r \delta_s = \frac{u_t}{y_t} y_o$ where $y_o = \beta_o P_s = \delta_s \geq 0$, we choose s by the following triple rule.

(12a) If any $\beta_t P_j > 0$, choose the smallest index s such that

$$\frac{y_o}{y_t} = \frac{\beta_o P_s}{\beta_t P_s} = \min_j \frac{\beta_o P_j}{\beta_t P_j} \;, \qquad \beta_t P_s > 0.$$

(12b) If all $\beta_t P_j = 0$, then u_t can never be reduced and there is no solution to (4). (U_t is independent of all P_j.)

(12c) If all $\beta_t P_j \leq 0$ and some $\beta_t P_j < 0$, then any allowable change in B at this point will not decrease u_t. In this case we seek to reduce the sum $\sum u_k$ of the u_k still formally in U. Forming the corresponding sum $\beta_\Sigma = \sum \beta_k$, we compute $\beta_\Sigma P_j$ for all $j > 0$. If any $\beta_\Sigma P_j > 0$, use (12a) to determine s (replacing $\beta_t P_j$ with $\beta_\Sigma P_j$). If all $\beta_\Sigma P_j \leq 0$, then $\sum u_k > 0$ is minimum, i.e., cannot be reduced without some x_j going negative and there is no feasible solution to (3),(4). (This is equivalent to the condition in the normal phase one when it is determined that no feasible solution exists.)

The reader will note that (12c) is nothing but a different way of operating the usual phase one. However, this rule does not come into play until all else has failed. Whatever novelty the present proposal can claim lies in this change of order and emphasis in tackling the four requirements (a), (b), (c), (d). Although the rules of selection for vectors to enter in and drop from the basis are more complicated here than in the normal procedure, this is a secondary consideration compared with time for and accuracy of

solution on a large electronic computer.

If (12) allows a P_s to be chosen, then as usual we represent it in terms of B

$$(13) \qquad Y_s = B^{-1}P_s = \{y_0, y_1, \ldots, y_m\} .$$

There is no guarantee that U_t can be eliminated and feasibility maintained even though (12a) was used. However, whatever change is made, P_s will enter B with its coefficient $x_s = \theta \geq 0$. The change in u_t (or $\sum u_k$ if (12c) was used) will be

$$u_t^* = u_t - \theta y_t \leq u_t$$

$$(\text{or } \sum u_k^* = \sum u_k - \theta \sum y_k \leq \sum u_k \text{ in case (12c)}) .$$

Again it should be emphasized that when Choice C must be used considerably, there is no guarantee that the present method will be any faster than the normal phase one. However, if there is a chance of short-cutting an unduly long number of iterations, the present method leaves the door open, as it were, for such a short cut to be taken.[4]

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[4] It is perhaps worth mentioning that when most of the a_{oj} are ≤ 0, Choice A is likely to be used from the outset and for many iterations. On the other hand, if most (or all) of the a_{oj} are ≥ 0, Choice C may have to be used at the outset and perhaps throughout the process and this involves more effort per iteration. Thus, the efficiency of this algorithm may well depend on the objectives of the model.

V-MODIFICATIONS OF THE SECOND SIMPLEX CRITERION

The second criterion in the simplex method is concerned with determining the index r of the basis vector to be replaced by P_s. Representing P_s in terms of B by (13), the usual criterion is to form the ratios (provided any $y_i > 0$)

(14) $\theta_i = \frac{v_i}{y_i}$ for all $y_i > 0$ and $i > 0$

and then to choose r by

(15) $\theta_r = \min \theta_i$ for all i included in (14),

whence $\theta_r \geq 0$. Providing <u>not all</u> $y_i \leq 0$ this same criterion will be used here.

In the usual procedure, the case of all $y_i \leq 0$ $(i > 0)$ implies that x_o has no finite maximum. For the present method, however, this last implication is no longer valid. It must be modified and for this purpose we distinguish three cases. We first note that all $y_i \leq 0$ can only occur in the case $\delta_s < 0$. (See Choice A in IV.)

Case I. All $u_i = 0$, all $y_i \leq 0$, and $y_k = 0$ for all u_k remaining in U. In this case a class of feasible solutions to (3),(4) can be constructed, whose values have no upper bound, namely,

(16) $B(V - \theta Y_s) + \theta P_s$

with a new value x_o^* given by (10). Since $y_o = \beta_o P_s = \delta_s < 0$, clearly

$$x_o^* \rightarrow +\infty \text{ as } \theta \rightarrow +\infty .$$

This case is equivalent to the case in the normal simplex method (phase two) where all $y_i \leq 0$ and the iterative procedure terminates.

The function of the redundant equation during phase two of the normal procedure is to insure that an artificial variable does not get above zero since this would require the redundant variable to go below zero (or vice versa) which is prohibited by (14) and (15). Since in the present method no redundant equation is used, this possibility must be avoided by examining the y_k for which u_k is in U. This leads us to case two.

Case II. All $u_i = 0$, all $y_i \leq 0$, but some $y_k < 0$ for u_k in U. Here P_s can be introduced into B by eliminating one of the U_k in B but not in P. For, since all $u_k = 0$, $\theta_r = {}^{u}r/y_r = 0$ $(y_r < 0)$ and the new solution vector V* will not differ from the present solution V (cf. (16)) and hence feasibility is maintained while one more u_k is discarded.

Case III. All $y_i \leq 0$ but some $u_k > 0$. (Again note that $\delta_s < 0$.) If $\theta_r > 0$, then x_o will increase but so will all u_k for which $y_k < 0$. Since there is no point in increasing x_o indefinitely at the expense of likewise increasing deviations from Q on the right-hand side, we arbitrarily reject the chosen P_s at this point and return to Choice C in section IV. In this case, Choice C may actually result in an increase in x_o, but for the present iteration we are concerned only with reducing $\sum u_k$ of the u_k still formally in U.

REFERENCES

[1] Dantzig, George B., Alex Orden, and Philip Wolfe, "The Generalized Simplex Method for Minimizing a Linear Form under Linear Inequality Restraints," RAND RM-1264, 5 April 1954 (Revised).

[2] ——————, ——————, "Duality Theorems," RAND RM-1265 30 October 1953 (Revised).

[3] ——————, "Computational Algorithm of the Simplex Method," RAND RM-1266, 26 October 1953 (Revised).

[4] ——————, "An Algebraic Proof of the MinMax Theorem," RAND RM-1267, (Revised) 18 December 1953.

[5] ——————, and Wm. Orchard-Hays, "Alternate Algorithm for the Revised Simplex Method using a product form for the inverse," RAND RM-1268, 19 November 1953.

[6] ——————, ——————, "The RAND Code for the Simplex Method (Preliminary information on Code OHSXI for the IBM 701 Electronic Computer)," RAND RM-1269 20 January 1954.

[7] ——————, "The Dual Simplex Algorithm," RAND RM-1270 3 May 1954 (Revised).

[8] ——————, "Composite Simplex—Dual Simplex Algorithm—I," RAND RM-1274, 26 April 1954.